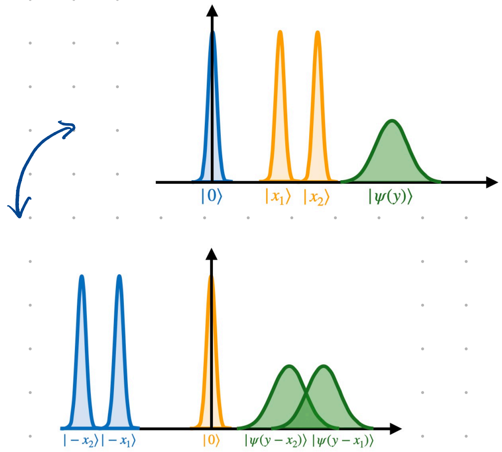
Non-ideal quantum reference frames

Outline

1. Ideal QRFs and how to go beyond them

Ideal QRFs



- quantum system carrying the regular representation U^{reg} of the symmetry group G
- Hilbert space $L^2(G, dg)$
- equivariant map $g \mapsto |g\rangle \in L^2(G)$ with perfectly distinguishable orthogonal frame orientation states

  $\langle g | g' \rangle = \delta(g, g')$

$\longrightarrow$ allows for quantum-controlled symmetry group transformations

for $G = \mathbb{R}(+)$ (one-dim translation group) and three systems A, B, C with $\mathcal{H}_A \cong \mathcal{H}_C \cong L^2(\mathbb{R})$ and B carrying a unitary rep U of G and U^{reg} acting on A and C

$$\hat{S}^{(A)\to(C)} = \hat{\mathcal{P}}_{AC}\, e^{-i/\hbar\, \hat{x}_C \hat{p}_B} \qquad \text{[Giacomini et al 2019]}$$

$$\stackrel{!}{=} \int_{\mathbb{R}} dx \; {}_A|-x\rangle\langle x|_C \otimes U(-x)_B \quad \text{with } U(y)_B = e^{-i/\hbar\, y \hat{p}_B}$$

or for general (locally compact) symmetry groups [de la Hamette, Galley 2020]

$$\hat{S}^{(A)\to(C)} = \int dg \; {}_A|g^{-1}\rangle\langle g|_C \otimes U(g^{-1})_B$$

$\hookrightarrow$ works for $\{|g\rangle\}_{g\in G}$ forming an ONB of $\mathcal{H}_A \cong \mathcal{H}_C \cong L^2(G)$

What if we loosen the assumption that the frame orientation states are orthogonal?

Theorem from [de la Hamette & Galley 2020]

> The QRF transformations of the perspectival approach for general (locally compact) symmetry groups are unitary
>
> if and only if the unitary rep. carried by the systems
>
> are the left and right regular reps. on states in the Hilbert space $L^2(G)$.

This means: QRF changes are unitary only if both frames are ideal frames where G acts regularly on itself and $\{|\psi(g)\rangle\}_{g\in G}$ are perfectly distinguishable.

Proof: Consider A, B, C with A and B acting as frames. Consider a left unitary representation $U_L(k)|g\rangle = |kg\rangle$.

Take two states relative to A:

$$|\psi^1\rangle = |e\rangle_A |g_1\rangle_B |g_2\rangle_C$$

$$|\psi^2\rangle = |e\rangle |h_1\rangle_B |g_2\rangle_C$$

as well as the linear superposition

$$|\psi^3\rangle = \alpha|\psi^1\rangle + \beta|\psi^2\rangle = |e\rangle_A (\alpha|g_1\rangle_B + \beta|h_1\rangle_B)|g_2\rangle_C$$

Let us now change to the perspective of B:

$$|\psi^1\rangle^{(A)} \xmapsto{\hat{S}} |\phi^1\rangle^{(B)} = |e\rangle_B |g_1^{-1}\rangle_A |g_2 g_1^{-1}\rangle_C$$

$$|\psi^2\rangle^{(A)} \xmapsto{\hat{S}} |\phi^2\rangle^{(B)} = |e\rangle_B |h_1^{-1}\rangle_A |g_2 h_1^{-1}\rangle_C$$

$$|\psi^3\rangle^{(A)} \xmapsto{\hat{S}} |\phi^3\rangle^{(B)} = |e\rangle_B (\alpha|g_1\rangle_A |g_2 g_1^{-1}\rangle_C + \beta|h_1^{-1}\rangle_A |g_2 h_1^{-1}\rangle_C)$$

The QRF change operator $\hat{S}^{(A)\to(B)}$ is unitary iff it preserves all inner products.

Let's consider the overlaps

$$\langle\psi^1|\psi^3\rangle = \alpha + \beta\langle g_1|h_1\rangle_B$$

and $\langle \phi^1 | \phi^3 \rangle = \alpha + \beta \langle \phi_1 | \phi_2 \rangle$

$= \alpha + \beta \langle g_1^{-1} | h_1^{-1} \rangle \underbrace{\langle g_2 g_1^{-1} | g_2 h_1^{-1} \rangle}_{= \langle g_1^{-1} | h_1^{-1} \rangle}$

$= \alpha + \beta (\langle g_1^{-1} | h_1^{-1} \rangle)^2$

"$\Rightarrow$" Assuming $\langle g_1 | h_1 \rangle = \langle g_1^{-1} | h_1^{-1} \rangle^2$

$\Leftrightarrow \langle g_1 | h_1 \rangle = \overline{\langle g_1^{-1} | h_1^{-1} \rangle}^2$

$\Leftrightarrow \langle g_1 | U_L(g_1^{-1})^+ U_L(g_1^{-1}) | h_1 \rangle = \overline{\langle h_1^{-1} | U_R(h_1^{-1})^+ U_R(h_1^{-1}) | g_1^{-1} \rangle}^2$

$\Leftrightarrow \langle e | g_1^{-1} h_1 \rangle = \overline{\langle e | g_1^{-1} h_1 \rangle}^2$ with $\begin{cases} U_L(g)|h\rangle = |gh\rangle \\ U_R(g)|h\rangle = |hg^{-1}\rangle \end{cases}$

$\rightarrow$ only holds when $g_1^{-1} h_1$ orthogonal to e

$\hat{S}$ unitary $\Rightarrow \langle g | h \rangle = \delta(g^{-1} h) \quad \forall g, h \in G$

"$\Leftarrow$" $\langle g | h \rangle = \delta(g^{-1} h) \ \forall g, h \in G \Rightarrow \hat{S}^{(i)\to(j)} = \int dg \, |g^{-1} \rangle\langle g|_j \otimes_k U(g^{-1})_k$ unitary

$\square$

<u>Conclusion</u>: If we want to construct unitary frame changes between non-ideal frames, we cannot <u>straightforwardly</u> do so in the perspectival framework (i.e. on the kinematical level).

$\hookrightarrow$ one way of allowing for unitary frame changes between non-ideal frames is to first remove the gauge-induced redundancy (cf. perspective-neutral approach)

2. Non-ideal QRFs in the perspective-neutral approach

Reference: [de la Hamette, Galley, Höhn, Loveridge, Müller, 2110.13824]

- Quantum system carrying a (strongly continuous) unitary representation U of the symmetry group G
- Different frame orientations are assigned overlapping, non-orthogonal coherent states: $\langle g | g' \rangle \neq \delta(g, g')$

2.1. Complete QRFs with coherent state systems

Consider a reference system R and a system of interest S, carrying a unitary tensor product representation of a (unimodular) Lie symmetry group G, acting on RS as

$$g \longmapsto U_{RS}(g) = U_R(g) \otimes U_R(g)$$

We can define globally invariant states (physical states)

$$|\psi_{phys}\rangle = U_{RS}(g)\, |\psi_{phys}\rangle \in \mathcal{H}_{phys}$$ physical Hilbert space

Physical states can be written as:

$$\hat{C}\, |\psi_{phys}\rangle = 0$$ (those states vanishing under the constraint)

or $$|\psi_{phys}\rangle := \Pi_{phys}\, |\psi_{kin}\rangle := \boxed{\int_G dg\, U_{RS}(g)\, |\psi_{kin}\rangle}$$

coherent G-twirling / coherent group averaging

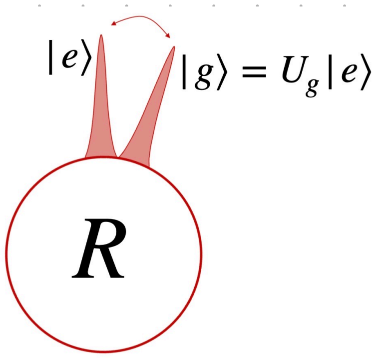
$\longrightarrow$ physical states by construction highly entangled states

We assume the existence of a coherent state system (i.e. system of generalised coherent states / Gilmore–Perelomov coherent states) (CSS)

$\{ U_R, |g\rangle_R \}$ such that G acts transitively on the CSS:

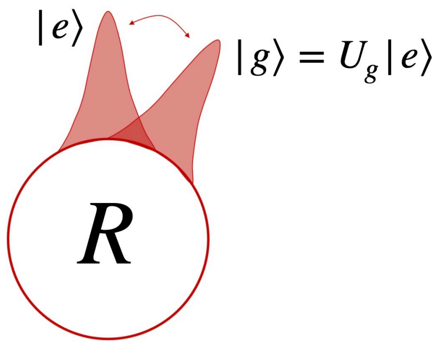
U_R: reference frame

Choose a normalised seed state $|e\rangle$, with e unit element of G, and set $|g\rangle := U_R(g)\, |e\rangle$.
Then generate $|gg'\rangle = U(g)|g'\rangle_R$.



Importantly, we <u>must</u> choose $|e\rangle$ in such a way that we obtain a

$$\boxed{\text{resolution of the identity: } \int_G dg\, |g\rangle\langle g|_R = c \cdot \mathbb{1}_R}$$



(for compact groups: $c > 0$, $c = \frac{1}{d_R}$, d_R: dimension of frame HS)

$\longrightarrow$ allows for the treatment of non-ideal frames

[dlH, Ludescher, Müller 2021]

The CSS defines a covariant POVM with elements $E_Y := \int_Y dg\, |g\rangle\langle g| \geq 0$
(Y : Borel set on the group)

- normalised : $E_G = \mathbb{1}_R$
- covariant : $E_{h \cdot Y} = U_h E_Y U_h^\dagger$

Now, we can write physical states as:

$$|\psi_{phys}\rangle = \int_G dg' \, |g'\rangle_R \otimes |\psi_S^{phys}(g')\rangle$$

Post-measurement state if we measure "orientation" g of R :

conditional / relational state

$$|\psi_S^{phys}(g)\rangle = \underbrace{\boxed{\mathcal{R}_S(g)}}_{\text{Schrödinger reduction map}} |\psi_{phys}\rangle = (\langle g|_R \otimes \mathbb{1}_S)\, |\psi_{phys}\rangle$$

↪ generalised Page - Wootters reduction of states

⚠ normalise states if possible

in fact, for compact G and $d_R < \infty$,

$$\mathcal{R}_S(g) := \sqrt{d_R}\, \langle g|_R \otimes \mathbb{1}_S$$

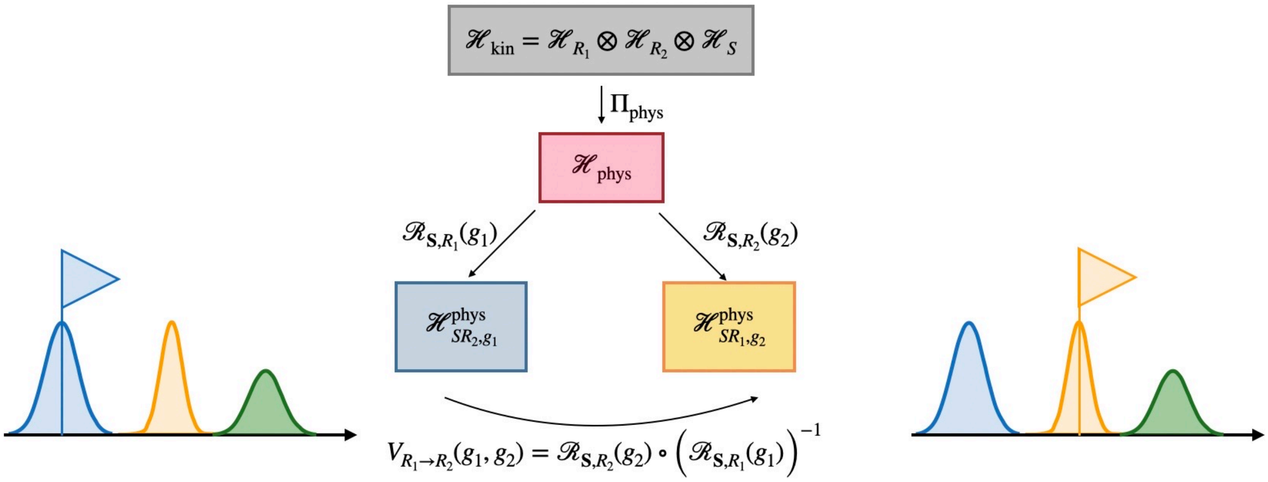
The conditional / relational states satisfy covariance property :

$$\begin{aligned}
\boxed{U_S(g')\, |\psi_S^{phys}(g)\rangle} &= (\langle g|_R \otimes \mathbb{1}_S)\, U_S(g')\, |\psi_{phys}\rangle \\
&= (\langle g|_R \otimes \mathbb{1}_S)(U_R^\dagger(g') \otimes \mathbb{1}_S)\, |\psi_{phys}\rangle \\
&= \langle g'g|_R\, |\psi_{phys}\rangle \\
&= \mathcal{R}_S(g'g)\, |\psi_{phys}\rangle \\
&= \boxed{|\psi_S^{phys}(g'g)\rangle}
\end{aligned}$$

The Schrödinger reduction maps are invertible, with inverse

$$\mathcal{R}_S^{-1}(g) := \Pi_{phys}\,(|g\rangle_R \otimes \mathbb{1}_S) = \int dg'\, |g'g\rangle_R \otimes U_S(g')$$

[Lemma 8, [2]]



QRF change /
quantum coordinate change

Quantum coordinate change:

$$\boxed{\begin{aligned} V_{R_i \to R_j}(g_i, g_j) &:= \mathcal{R}^{R_j}_{S}(g_j) \circ \mathcal{R}^{R_i}_{S}(g_i)^{-1} \\ V_{R_i \to R_j}(g_i, g_j)\, |\psi^{\text{phys}}_{R_j S}(g_j)\rangle &= |\psi^{\text{phys}}_{R_i S}(g_i)\rangle \end{aligned}}$$

For $i \neq j$, $V_{R_i \to R_j}(g_i, g_j) = \int_G dg\, |g g_i\rangle_{R_i} \otimes \langle g^{-1} g_j|_{R_j} \otimes U_S(g)$

(Theorem 4, [2])

Proof: $V_{R_i \to R_j}(g_i, g_j) = \left(\langle g_j|_{R_j} \otimes \mathbb{1}_{R_i S}\right) \underline{\Pi_{\text{phys}}} \left(|g_i\rangle_{R_i} \otimes \mathbb{1}_{R_j S}\right)$

$= \int dg\, U_{R_1}(g) \otimes U_{R_2}(g) \otimes U_S(g)$

$\square$

Relational Dirac observables

f_S arbitrary system observable in $\mathcal{A}_S := \mathcal{L}(\mathcal{H}_S)$

$|g\rangle\langle g|_R \otimes f_S \in \mathcal{A}_{kin}$: non-invariant description of system property f_S conditional on R being in orientation $g \in G$

Applying the G-twirl:

$$\boxed{F_{f_S,R}(g) := \mathcal{G}\left(|g\rangle\langle g|_R \otimes f_S\right)}$$

$$\approx \Pi_{phys}\left(|g\rangle\langle g|_R \otimes f_S\right)$$

$$= \int_G dg' \, |g'g\rangle\langle g'g|_R \otimes U_S(g')\, f_S\, U_S(g')^\dagger$$

$$= \int dg' \, |g'\rangle\langle g'|_R \otimes U_S(g'g^{-1})\, f_S\, U_S(g'g^{-1})^\dagger$$

↪ invariant encoding of system property f_S conditional on R being in orientation $g \in G$

↪ covariant POVM construction

Example: translation group $G = (\mathbb{R}, +)$, $\mathcal{H}_{kin} = L^2(\mathbb{R}) \otimes L^2(\mathbb{R})$

$\hat{q}_R, \hat{q}_S$: position operators of R and S

relational observable encoding position of S conditioned on R being in orientation (position) y:

$$F_{\hat{q}_S,R}(y) = \int dx \, |x\rangle\langle x|_R \otimes e^{-i(x-y)\hat{p}_S}\, \hat{q}_S\, e^{+i(x-y)\hat{p}_S}$$

$$= \int dx \, |x\rangle\langle x|_R \otimes (\hat{q}_S + x - y)$$

$$= \hat{q}_S - \hat{q}_R + y$$

The relational observables define an algebra homomorphism from $\mathcal{A}_S^{phys} \subset \mathcal{A}_S$ of system observables to physical observable algebra $\mathcal{A}^{phys}$ of Dirac observables on $\mathcal{H}_{phys}$.

consider
$$\Pi_S^{phys}(g) := \left(\langle g|_R \otimes \mathbb{1}_S\right) \Pi_{phys} \left(|g\rangle_R \otimes \mathbb{1}_S\right)$$
$$= \int dg' \, \langle g|g'g\rangle_R \, U_S(g')$$
$$= \left(\langle e|_R U_R^\dagger(g) \otimes \mathbb{1}_S\right) \Pi_{phys} \left(U_R(g)|e\rangle_R \otimes \mathbb{1}_S\right)$$
$$= U_S(g) \, \Pi_S^{phys}(e) \, U_S^\dagger(g)$$

$$\longrightarrow \boxed{f_{S,g}^{phys} := \Pi_S^{phys}(g) \, f_S \, \Pi_S^{phys}(g)} \in \mathcal{A}_{S,g}^{phys} \quad \text{for arbitrary } f_S \in \mathcal{A}_S$$

Quantum coordinate changes

$$\Lambda_{R_i \to R_j}(g_i, g_j) : \mathcal{A}_{R_j S, g_i}^{phys} \longrightarrow \mathcal{A}_{R_i S, g_j}^{phys}$$
$$\Lambda_{R_i \to R_j}(g_i, g_j)\left(f_{R_j S, g_i}^{phys}\right) = V_{R_i \to R_j}(g_i, g_j) \, f_{R_j S, g_i}^{phys} \, V_{R_i \to R_j}(g_i, g_j)$$

2.2. Equivalence of perspectival and PN approaches for ideal QRFs

For $L^2(G)$ w/ regular representation, with $g_i = g_j = e$

$$|e\rangle \otimes |\psi^{phys}_{R_i,S}(e)\rangle = V_{R_i \to R_j}(e,e)\, |e\rangle \otimes |\psi^{phys}_{R_jS}(e)\rangle$$

Remember:

$$V_{R_i \to R_j}(g_i, g_j) = \int_G dg\, |g g_i\rangle_{R_i} \otimes \langle g^{-1} g_j|_{R_j} \otimes U_S(g)$$

$$\longrightarrow V_{R_i \to R_j}(e,e) = \int_G dg\, {}_{R_i}|g\rangle\langle g^{-1}|_{R_j} \otimes U_S(g)$$

$$= \int_G dg\, {}_{R_i}|g^{-1}\rangle\langle g|_{R_j} \otimes U_S(g^{-1}) \quad ✓$$

<u>Conclusion</u>: The perspectival and the perspective-neutral approaches are equivalent for ideal QRFs, i.e. $L^2(G)$ frames with orthogonal frame orientation states.

The frame change maps are not applied to the same set of states in the two approaches:

→ in the perspectival approach, the transformations are applied to the kinematical states of the form $|e\rangle_{R_i} \otimes |\psi\rangle_{R_jS}$ (called alignable states) with $|\psi\rangle_{R_jS} \in \mathcal{H}_{R_j} \otimes \mathcal{H}_S$

→ in the perspective-neutral approach the transformations are applied to states $|\psi^{phys}_{R_jS}\rangle$ in the physical system Hilbert space $\mathcal{H}^{phys}_{R_jS,S}$ generally a strict subspace of $\mathcal{H}_{R_j} \otimes \mathcal{H}_S$

→ $\mathcal{H}_{R_j} \otimes \mathcal{H}_S = \mathcal{H}^{phys}_{R_jS,S} \quad \forall g \in G$ only for ideal reference frames

2.3. Classification of non-ideal frames (non-exhaustive)

- Complete QRFs with coherent state systems

  ≅ action of G on G by left multiplication

  * G acts regularly on the CSS (regular action, ~~not~~ not to be confused with regular representation)

    regular action = free and transitive action

    where free action: $g \cdot x = x$ for some $x \in X$ implies $g = e$

    transitive action: $\forall x, y \in X \; \exists g \in G$ s.t. $g \cdot x = y$

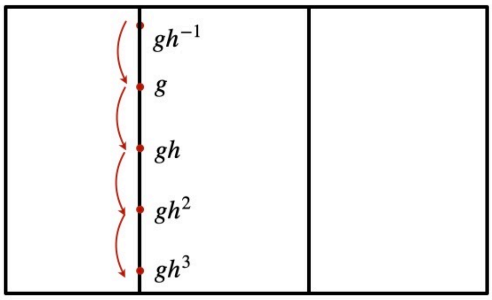
  * "complete" because the frame orientation states can be used to completely parametrise the G orbits in Hilbert space or the observable algebra

  * trivial stabiliser subgroup

- Incomplete QRFs: non-trivial stabiliser group

  QRF with non-trivial stabiliser subgroup

  $$H = \{ g \in G \text{ s.t. } |g X g| = |e X e| \}$$



$$F_{f_S,R}(g) = \int_X d_X x \, |g_x g\rangle\langle g_x g|_R \otimes U_S(g_x) \left(\int_H d_H h \, U_S(h) \, f_S \, U_S^\dagger(h) \right) U_S^\dagger(g_x) \, .$$

$$f_S \sim f'_S \quad \Leftrightarrow \quad f_S^{\text{avg}} := \frac{1}{\text{Vol}(H)} \int_H d_H h \, U_S(h) \, f_S \, U_S^\dagger(h) = \frac{1}{\text{Vol}(H)} \int_H d_H h \, U_S(h) \, f'_S \, U_S^\dagger(h) \, .$$

- Overcomplete QRFs

  ↳ frames whose orientation states G acts on freely but not transitively

- QRFs based on quantum groups (not treated in PN paper) (see e.g. 2504.00569)

2.4. Example for non-ideal (complete) QRFs: the group S_3

[work with Stefan Ludescher and Markus Müller]

S_3 has 6 elements $\{e, d, d^2, s_1, s_2, s_3\}$ and is the smallest non-Abelian group up to isomorphisms.

Cayley table for S_3:

$\cdot$	e	d	d^2	s_1	s_2	s_3
e	e	d	d^2	s_1	s_2	s_3
d	d	d^2	e	s_3	s_1	s_2
d^2	d^2	e	d	s_2	s_3	s_1
s_1	s_1	s_2	s_3	e	d	d^2
s_2	s_2	s_3	s_1	d^2	e	d
s_3	s_3	s_1	s_2	d	d^2	e

3 conjugacy classes $[e] = \{e\}$, $[d] = \{d, d^2\}$, $[s_1] = \{s_1, s_2, s_3\}$

$\hookrightarrow$ 3 irreducible representations:

- the trivial representation $\mathbb{1}$ (1d)
- the sign representation Π_2 (1d)
- the standard representation Π_3 (2d)

Coherent state system

Our Hilbert space of interest is $\mathcal{H} = \mathbb{C}^3$ (notice that $\dim(\mathcal{H}) = 3 < 6 = |S_3|$) for one system and

$$\mathcal{H}^N = (\mathbb{C}^3)^{\otimes N} \text{ for } N \text{ systems.}$$

We consider the (unitary) representation of S_3 on $\mathbb{C}^3$ where S_3 is represented as permutations of the three basis vectors $\{|0\rangle, |1\rangle, |2\rangle\}$.
Write $U(g) = U_g$ for every $g \in S_3$.

In contrast to working with Hilbert space $L^2(S_3)$, it is not possible to find a set of vectors $\{|g\rangle\}_{g\in S_3}$ s.t. $\langle g|g'\rangle = \delta_{gg'}$.
It is, however, possible to find a set of vectors $\{|g\rangle\}_{g\in S_3}$ s.t.

- $|g\rangle = U(g)|e\rangle$ (1)
- $|g\rangle \neq |g'\rangle$ for $g \neq g'$ (2)
- $\sum_{g\in S_3} |g\rangle\langle g| = k \cdot \mathbb{1}$ $(k \in \mathbb{C})$ (3)

Write the seed state $|e\rangle$ in the basis $\{|0\rangle, |1\rangle, |2\rangle\}$ as

$$|e\rangle = \alpha|0\rangle + \beta|1\rangle + \gamma|2\rangle.$$

$$(1) \Rightarrow \begin{cases} |d\rangle = \alpha|1\rangle + \beta|2\rangle + \gamma|0\rangle \\ |d^2\rangle = \alpha|2\rangle + \beta|0\rangle + \gamma|1\rangle \\ |s_1\rangle = \alpha|0\rangle + \beta|2\rangle + \gamma|1\rangle \\ |s_2\rangle = \alpha|2\rangle + \beta|1\rangle + \gamma|0\rangle \\ |s_3\rangle = \alpha|1\rangle + \beta|0\rangle + \gamma|2\rangle \end{cases}$$

$(2) \Rightarrow \alpha \neq \beta, \ \alpha \neq \gamma, \ \beta \neq \gamma$

Write $|g\rangle = \alpha|a\rangle + \beta|b\rangle + \gamma|c\rangle$ where $a, b, c \in \{i\}_{i=0}^{2}$, $a \neq b, b \neq c, a \neq c$

$$(8) \Rightarrow \begin{cases} |\alpha|^2 + |\beta|^2 + |\gamma|^2 = \frac{k}{2} \\ \alpha\beta^* + \alpha^*\beta + \alpha^*\gamma + \alpha\gamma^* + \beta\gamma^* + \beta^*\gamma = 0 \end{cases}$$

Example of a valid choice: $|e\rangle = \frac{1}{\sqrt{2}}|0\rangle + \frac{i}{\sqrt{2}}|1\rangle \ (+0|2\rangle)$ $\quad (\rightarrow k=2)$

The physical Hilbert space

$$\mathcal{H}_{phys} = \{ |\phi\rangle \in \mathcal{H}^N \mid U(g)^{\otimes N} |\phi\rangle = |\phi\rangle \ \ \forall g \in S_3 \}$$

Note that every $|\phi\rangle$ defines a subspace $\mathcal{H}_\phi = \text{span}\{|\phi\rangle\}$ on which $\mathcal{H}^{\otimes N}$ acts trivially

$\rightarrow$ have to find the decomposition of U_g into irreps, and then we can associate the copies of the trivial representation with $\mathcal{H}_{phys}$.

$N=1$:
$$|\mathbb{1}\rangle = \frac{1}{\sqrt{3}}(|0\rangle + |1\rangle + |2\rangle)$$
$$|\pi_3, \mp\rangle = \frac{1}{\sqrt{3}}\left(|0\rangle + e^{\pm\frac{2\pi i}{3}}|1\rangle + e^{\mp\frac{2\pi i}{3}}|2\rangle\right)$$

$$U_g = \mathbb{1} \oplus \pi_3$$

standard representation

trivial representation

$\hookrightarrow$ for $N=1$, $\mathcal{H}_{phys} = \text{span}\{|\mathbb{1}\rangle\} = \text{span}\{\frac{1}{\sqrt{3}}(|0\rangle + |1\rangle + |2\rangle)\}$

Lemma: For N systems,

$$U_g^{\otimes N} = \frac{3^{N-1}+1}{2}\,\mathbb{1} \oplus \frac{3^{N-1}-1}{2}\,\pi_2(g) \oplus 3^{N-1}\,\pi_3(g)$$

Proof by induction. (Hint: prove that $\pi_2 \otimes \pi_3 = \pi_3$
$\pi_3 \otimes \pi_3 = \mathbb{1} \oplus \pi_2 \oplus \pi_3$)

Corollary: $\dim(\mathcal{H}_{phys}) = \frac{3^{N-1}+1}{2}$

Concrete example

[Appendix D, [Cepollaro, Akil, Cieslinski, ACdlH, Brukner, PRL 2025]

Consider $N=3$, denote the systems by A, B, C, $\mathcal{H}_{kin} = (\mathbb{C}^3)^{\otimes 3}$

$\rightarrow \dim \mathcal{H}_{phys} = 5$

Consider the following kinematical state:

$$|\psi_{kin}\rangle_{CAB} = |0\rangle_C \otimes \frac{1}{\sqrt{3}} \left(i|0\rangle_A + |1\rangle_A - |2\rangle_A \right) \otimes |2\rangle_B$$

$$|\psi_{phys}\rangle_{CAB} = \frac{1}{\sqrt{|S_3|}} \sum_{g \in S_3} U_{CAB}(g)$$

$$\hookrightarrow |\psi_{\text{phys}}\rangle_{CAB} = \frac{1}{\sqrt{6}}\frac{1}{\sqrt{3}}\big(i|002\rangle + |012\rangle - |022\rangle + i|110\rangle + |120\rangle - |100\rangle + i|221\rangle + |201\rangle - |211\rangle$$
$$+ i|001\rangle + |021\rangle - |011\rangle + i|220\rangle + |210\rangle - |200\rangle + i|112\rangle + |102\rangle - |122\rangle\big)_{CAB}.$$

Choosing the valid seed state $|e\rangle = \frac{1}{\sqrt{2}}(|0\rangle + i|1\rangle)$, the corresponding Schrödinger reduction map is

$$\mathcal{R}^{C}_{S}(e) = \sqrt{3}\,\frac{1}{\sqrt{2}}\left(\langle 0| - i\langle 1|\right)_C \otimes \mathbb{1}_{AB}$$

reduced / conditional states:

$$\left.\begin{aligned}
\mathcal{R}_{\mathbf{S},C}(e)|\psi_{\text{phys}}\rangle_{CAB} &= \frac{\sqrt{3}}{6}\big(i|00\rangle + i|01\rangle + |10\rangle - |11\rangle + 2|12\rangle - i|20\rangle + |21\rangle + (i-1)|22\rangle\big)_{AB}.\\
\mathcal{R}_{\mathbf{S},A}(e)|\psi_{\text{phys}}\rangle_{CAB} &= \frac{\sqrt{3}}{6}\big(2i|01\rangle + (-1-i)|20\rangle + (1+i)|21\rangle + 2|12\rangle\big)_{CB}.
\end{aligned}\right\} \text{satisfy covariance property}$$