MOTIVATION

Reference frames and coordinate systems are essential in physics: provide the point of view from which observations are carried out.

Not exactly the same, definitions in the literature differ.

For us: set of variables associated to a PHYSICAL SYSTEM, that we can use to describe other physical systems relationally.

Since 1967: QRFs in QG (DeWitt) / QI (Aharonov, Susskind)

Necessity of considering RFs associated to physical systems.

$\boxed{\text{QI}}$ A and B, they need to share information but do not know how they are oriented relative to each other; they want to identify the "z" direction with a big spin, which has some quantum uncertainty associated to it.

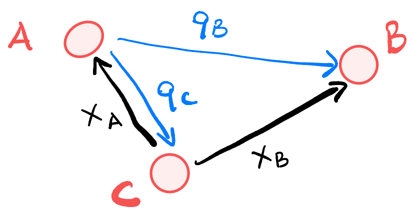
$\boxed{\text{QG}}$ If gravity is quantum $\Rightarrow$ no spacetime $\Rightarrow$ no classical RF.
In GR it does not make sense to localise a single system
$\Rightarrow$ I need a physical system to localise another physical system.
Need to consider RELATIVE localisation between two material quantum systems in whatever replaces spacetime
(Also more technical reasons related to the quantisation of gravity as a gauge theory, see Daniele's lectures)

One particular approach to QRFs: PERSPECTIVAL / FOUNDATIONAL

Research programme: many open questions (e.g. how to include the observer, bridge with gravity / QI, etc...)

In recent years the approaches started to come together.

GIACOMINI, CASTRO-RUIZ, BRUKNER, NAT. COMMUN (2019)



C: initial frame
A: final frame
B: quantum system

$x_B \to q_B - q_C$
$x_A \to -q_C$

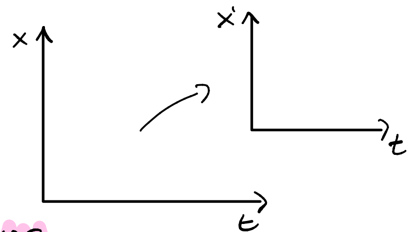
No ext. frame → only rel. pos. are phys.

When I stand in C, I describe A and B
When I stand in A, I describe B and C
} positions already imply specifying a RF

Reference frame transformations in QM

Transl. $U_T = e^{-\frac{i}{\hbar} x_0 \hat{p}}$

Boost: $U_G = e^{\frac{i}{\hbar} v_0 \hat{G}}$, $\hat{G} = \hat{p}t - m\hat{x}$



Class of interesting transf.
⇒ EXTENDED GALILEAN TRANSFORMATIONS
(Greenberger, 1979)

$$x' = x - \xi(t)$$

$$\hat{U}(t) = e^{-\frac{i}{\hbar} m \dot{\xi}(t) \hat{x}} \, e^{\frac{i}{\hbar} \xi(t) \hat{p}} \, e^{-\frac{i}{\hbar} \frac{m}{2} \int_0^t ds\, \dot{\xi}^2(s)}$$

Starting from Schrödinger equation $\quad i\hbar \frac{d\hat{e}}{dt} = [\hat{H}, \hat{e}] \quad (*)$
then we say that the physical laws are COVARIANT if

$$\hat{e} \to \hat{U}(t)\, \hat{e}\, \hat{U}^+(t) = \hat{e}' \; ; \quad (*) \to i\hbar \frac{d\hat{e}'}{dt} = [\hat{H}', \hat{e}']$$

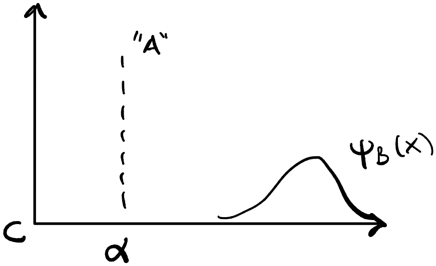
$$\hat{H}' = \hat{U}(t) \hat{H} \hat{U}^+(t) + i\hbar \frac{d\hat{U}}{dt} \hat{U}^+(t)$$

If $\hat{H} = \frac{\hat{p}^2}{2m}$

$$\Rightarrow \hat{H}' = \frac{\hat{p}^2}{2m} + m \ddot{\xi}(t) \hat{x}$$

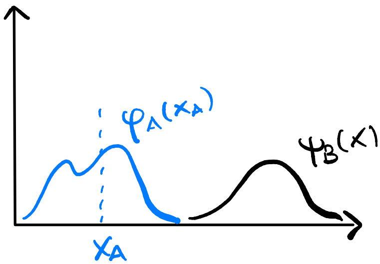
$\boxed{\text{SYMMETRY} \quad \hat{H}' = \hat{H}}$

N.B.: ξ is a classical function ⇒ rel. between old and new RF!



TRANSLATION

$$e^{\frac{i}{\hbar}\alpha \hat{p}_B} |\psi\rangle_B = \int dx_B\, \psi_B(x_B)\, |x_B-\alpha\rangle_B$$



$$\int dx_A\, \varphi_A(x_A)\, e^{\frac{i}{\hbar} x_A \hat{p}_B} |\psi\rangle_B |x_A\rangle_A = e^{\frac{i}{\hbar}\hat{x}_A \hat{p}_B} |\varphi\rangle_A |\psi\rangle_B$$

1. Add Hilb. sp. for A
2. Promote variable to operator
   $\alpha \to \hat{x}_A$

+ PARITY-SWAP
$$\mathcal{P}_{AC}|x\rangle_A = |-x\rangle_C$$
$$\mathcal{P}_{AC}|p\rangle_A = |-p\rangle_C$$
$$\Rightarrow \hat{S}_x = \mathcal{P}_{AC}\, e^{\frac{i}{\hbar}\hat{x}_A \hat{p}_B}$$

SIMPLEST QRF TRANSF.!

EXAMPLES

1) $|x_0\rangle_A |\psi\rangle_B \to |-x_0\rangle_C\, e^{\frac{i}{\hbar}x_0 \hat{p}_B} |\psi\rangle_B$

2) $\frac{1}{\sqrt{2}}\left(|x_1\rangle_A + |x_2\rangle_A\right)|\psi\rangle_B \to \frac{1}{\sqrt{2}}\left(|-x_1\rangle_C\, e^{\frac{i}{\hbar}x_1 \hat{p}_B}|\psi\rangle_B + |-x_2\rangle_C\, e^{\frac{i}{\hbar}x_2 \hat{p}_B}|\psi\rangle_B\right)$

3) $\frac{1}{\sqrt{2}}\left(|x_1\rangle_A |x_1+L\rangle_B + |x_2\rangle_B |x_2+L\rangle_B\right) \to \frac{1}{\sqrt{2}}|L\rangle_B\left(|-x_1\rangle_C + |-x_2\rangle_C\right)$

$\Rightarrow$ Entanglement + superposition are QRF dep.!

Dynamical laws

The QRF also has a Ham. Assume $\hat S$ unitary

$$H^{(C)}_{AB}\,,\ e^{(C)}_{AB} \quad \Rightarrow \quad i\hbar\,\frac{d\hat e^{(C)}_{AB}}{dt} = [\hat H^{(C)}_{AB},\ \hat e^{(C)}_{AB}]$$

$$\hat e^{(A)}_{BC} = \hat S\,\hat e^{(C)}_{AB}\,\hat S^{\dagger}$$

$$\hat H^{(A)}_{BC} = \hat S\,\hat H^{(C)}_{AB}\,\hat S^{\dagger} + i\hbar\,\frac{d\hat S}{dt}\,\hat S^{\dagger}$$

$\hat S$ is a symmetry if $\hat H^{(A)}_{BC}$ has the same functional form as $\hat H^{(C)}_{AB}$ but with all labels $A \leftrightarrow C$

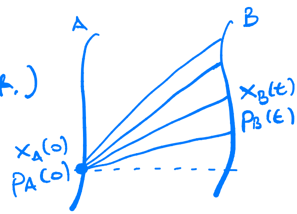
In particular, we have sep. of EXT. GAL. TRANSF.

$$\hat S_i = e^{-\frac{i}{\hbar}H^{(A)}_C t}\ \underbrace{\hat{\mathcal{P}}_{AC}\left(e^{\frac{i}{\hbar}\ln\sqrt{\frac{m_C}{m_A}}\,\hat D_A}\right)}_{\mathcal{P}^{(\sigma)}_{AC}}\ \hat U_i(t)\ e^{\frac{i}{\hbar}\hat H_A t}$$

$\hat U_i(t)$: classical RF transf where parameters → operators on A

$\hat D_A = \hat x_A \hat p_A + \hat p_A \hat x_A$: only necessary in some cases (boosts + accel.)

$$\mathcal{P}^{(\sigma)}_{AC}\,\hat p_A\,\mathcal{P}^{\dagger(\sigma)}_{AC} = -\frac{m_A}{m_C}\,\hat\pi_C$$ ⇒ momenta transf. in such a way that they are easily connected to the rel. vel.



GIVE INTUITION ABOUT TRANSF. (especially Hamilt.)

EX: $$\hat S_b = e^{-\frac{i}{\hbar}\frac{\hat\pi_C^2}{2m_C}t}\ \mathcal{P}^{(\sigma)}_{AC}\ e^{\frac{i}{\hbar}\frac{\hat p_A}{m_A}\hat G_B}\ e^{+\frac{i}{\hbar}\frac{\hat p_A^2}{2m_A}t}$$

$$H^{(C)}_{AB} = \frac{\hat p_A^2}{2m_A} + \frac{\hat p_B^2}{2m_B}$$

$$\rightarrow\ H^{(A)}_{BC} = \frac{\hat\pi_B^2}{2m_B} + \frac{\hat\pi_C^2}{2m_C}$$ SYMMETRY!

QRF transf. may seem strange, but they form a group!

BALLESTEROS, GIACOMINI, GUBITOSI QUANTUM (2021)

$\hat{P}_{AB} = \hat{x}_A \otimes \hat{p}_B$; $\hat{K}_{AB} = \frac{\hat{p}_A}{m_A} \otimes \hat{G}_B$; $\hat{D}_{A|B} = \frac{1}{2}(\hat{x}_A \hat{p}_A + \hat{p}_A \hat{x}_A)_{|B}$, ecc.

$[\hat{A}_i , \hat{A}_j] = f_{ijk} \hat{A}_k$ → group has 7 generators, not a known group

Important because I can compose transf. and remain in the framework

What are QRFs useful for?

- Define rest frame of a quantum system
  - * Stern-Gerlach FG, CASTRO-RUIZ, BRUKNER, PRL (2019)
  - * Rel. Bell ineq. STREITER, FG, BRUKNER, PRL (2021)
  - * Doppler effect FG, CASTRO-RUIZ, BRUKNER, NAT. COMM. (2019)
  - * Decay rate FG, KEMPF, PRD (2022)

- Non-classical spacetime ⟿ see other lectures!

PERSPECTIVE NEUTRAL APPROACH TO QRFs

VANRIETVELDE, HÖHN, GIACOMINI, CASTRO-RUIZ, QUANTUM (2020)

Inspired to GR. There is a description with no physical counterpart, but that can be mapped to one in which only relational d.o.f. are meaningful.
Redundancy ~ diffeo invariance in GR.

N particle Lagrangian ($m_i = 1$)

$$\mathcal{L} = \frac{1}{2}\sum_i \dot{q}_i^2 - \underbrace{\frac{1}{2N}\left(\sum_i \dot{q}_i\right)^2}_{E^K_{CM}} - V(\{q_i - q_j\})$$

E^K_{CM} : only the motion rel. to CM contributes

$$\frac{\partial \mathcal{L}}{\partial \dot{q}_i} = \dot{q}_i - \frac{1}{N}\sum_j \dot{q}_j = p_i \implies \boxed{\sum_i p_i = 0 \quad \text{CM momentum is zero} \;\; \hookrightarrow \text{dynamical constraint}}$$

REDUNDANCY

EOM $\quad \ddot{q}_i - \frac{1}{N}\sum_j \ddot{q}_j + \frac{\partial V}{\partial q_i} = 0$

Consistency check: $\sum_i \ddot{q}_i - \sum_j \ddot{q}_j + \sum_i \frac{\partial V}{\partial q_i} = \sum_i \frac{\partial V}{\partial q_i} = 0$ always true for a central potential

If we calculate the Ham. we find (Legendre transf.)

$H = \sum_i \frac{p_i^2}{2} + V(\dots)$ but we know that, on the EOM $\sum_i p_i = 0$

Hence I can write

$\mathcal{C}$: CONSTRAINT

$$H = \sum_i \frac{p_i^2}{2} + V(\dots) + \lambda\,(\sum_i p_i) \implies \text{does not change the physics}$$

$\hookrightarrow$ LAGRANGE MULTIPLIER

On the solutions of the EOM $\quad \mathcal{C} \approx 0$ (weak equality)

Treat λ as a dynamical variable. H is indep. of p_λ, hence

$\dot{\lambda} = \frac{\partial H}{\partial p_\lambda} = 0$ (λ does not evolve). We also have

$\dot{p}_\lambda = \{p_\lambda, H\} = -\mathcal{C} = 0$ ✓ because $p_\lambda(\lambda, \dot{\lambda})$

Finally, $\mathcal{E}$ has to be preserved by the dynamics

$\dot{\mathcal{E}} = \{\mathcal{E}, H\} = 0$ by construction in our case

If this does not happen, we need to add whatever we find as an additional constraint until we find no contradiction.

DIRAC QUANTISATION

$q_i \to \hat{q}_i$; $p_i \to \hat{p}_i$; $[\hat{q}_i, \hat{p}_j] = i\hbar\, \delta_{ij}$

$\Rightarrow$ Then enforce the constraint.

$$\hat{H} = \Sigma_i \frac{\hat{p}_i^2}{2} + V(\{\hat{q}_i - \hat{q}_j\}) + \lambda \hat{\mathcal{E}} \quad , \quad \hat{\mathcal{E}} = \Sigma_i \hat{p}_i$$

We know that $\hat{\mathcal{E}}$ vanishes on the solutions of the EOM. What does it mean in QT?

Only allowed states are those s.t. $\hat{\mathcal{E}} |\Psi\rangle_{ph} = 0$

We have two Hilb. sp. $\mathcal{H}_{kin} \to$ all states

$\mathcal{H}_{ph} \to$ only states s.t. $\hat{\mathcal{E}} |\Psi\rangle_{ph} = 0$

Each Hilb. sp. is endowed with a scalar product, in principle diff.!

$\Sigma_i \hat{p}_i$ has continuous spectrum, in part. around zero

$\Rightarrow$ we have to find a new scalar product that makes the states well-normalised on H_{phys}

IMPROPER PROJECTOR $\quad \delta(\hat{P}) = \frac{1}{2\pi} \int ds\, e^{is\hat{P}} \quad \hat{P} = \Sigma_i \hat{p}_i$

$\delta(\hat{P}) : \mathcal{H}_{kin} \to \mathcal{H}_{phys}$

$$\langle \psi | \varphi \rangle_{phys} = \langle \psi | \delta(\hat{P}) | \varphi \rangle_{kin}$$

How do I reduce to the persp. of a specific QRF?

Assume 3 particles A, B, C.

• Physical state $\quad \delta(\hat{P})\,|\Psi\rangle^{kin}_{ABC} = |\psi\rangle_{phys}$

$$= \int dp_A\, dp_B\, dp_C\ \delta(p_A+p_B+p_C)\,\Psi_{kin}(p_A, p_B, p_C)\,|p_A\rangle_A |p_B\rangle_B |p_C\rangle_C =$$

$$= \int dp_B\, dp_C\ \Psi_{kin}(-p_B-p_C, p_B, p_C)\,|-p_B-p_C\rangle_A\, |p_B\rangle_B |p_C\rangle_C$$

↳ not the rel. st. !

TRIVIALISATION OPERATOR $\hat{T}_A$ s.t.

$$\hat{T}_A(\hat{q}_{B|C} - \hat{q}_A)\hat{T}_A^{\dagger} = \hat{q}_{B|C} \quad ; \quad \hat{T}_A\, \hat{p}_{B|C}\, \hat{T}_A^{\dagger} = \hat{p}_{B|C}$$

$\hat{T}_A(\hat{p}_A + \hat{p}_B + \hat{p}_C)\hat{T}_A^{\dagger} = \hat{p}_A \Rightarrow$ shift all the redundancy on A to remove it from the description

$$\hat{T}_A = e^{\frac{i}{\hbar}\hat{q}_A(\hat{p}_B + \hat{p}_C)}$$

$$\hat{T}_A\,|\Psi\rangle_{phys} = \underbrace{|p_A = 0\rangle_A}_{\text{need to remove}} \int dp_B\, dp_C\ \Psi_{kin}(-p_B-p_C, p_B, p_C)\,|p_B\rangle_B |p_C\rangle_C$$

Project on $\int dp'\ {}_A\langle p'| = 2\pi\langle q_A = 0| \qquad \int dp' \langle p'|p_A = 0\rangle = 1$

Physical meaning: "force" the QRF A to be in the origin!

$$|\psi\rangle^{(A)}_{BC} = \int dp_B\, dp_C\ \Psi_{kin}(-p_B-p_C, p_B, p_C)\,|p_B\rangle_B |p_C\rangle_C$$

↳ relational state!

$$\begin{tikzcd}
 & \boxed{\begin{matrix}\text{PERSP. NEUTR.}\\ \hat{H} = \sum_i \frac{\hat{p}_i^2}{2} + V(..) + \lambda\hat{\mathcal{E}}\end{matrix}}\ \mathcal{H}_{kin} \arrow[d, "\delta(\hat{P})"] & \\
|\psi\rangle^{(C)}_{AB} \arrow[rr, dashed, bend right, "?"'] & \mathcal{H}_{phys} \arrow[l, "2\pi\langle q_C = 0|\hat{T}_C"'] \arrow[r, "2\pi\langle q_A = 0|\hat{T}_A"] & |\psi\rangle^{(A)}_{BC}
\end{tikzcd}$$

Answer: $\hat{S}_x = \mathcal{P}_{AC}\, e^{\frac{i}{\hbar}\hat{x}_A \hat{p}_B}$!